Chapter 5 Duality Theory

- Associated with every linear program is another called its dual.
- The dual of this dual linear program is the original program.
- Every feasible solution for one of the two programs gives a bound on the optimal objective value for the other.

1. Motivation - Finding Upper Bounds

Example

maximize \( 4x_1 + x_2 + 3x_3 \)
subject to \( x_1 + 4x_2 \leq 1 \)
\( 3x_1 - x_2 + x_3 \leq 3 \)
\( x_1, x_2, x_3 \geq 0. \)

Let \( \zeta^* \) be the optimal objective function value.

Every feasible solution provides a lower bound on \( \zeta^* \),
e.g. \((x_1,x_2,x_3) = (1,0,0)\) tells us that \( \zeta^* \geq 4 \).

\((x_1,x_2,x_3) = (0,0,3)\) tells us that \( \zeta^* \geq 9 \). (How good is it?)
The sum of the multiplication of the first constraint by 2 and that of the second constraint by 3:

\[
\begin{align*}
2 \left( x_1 + 4x_2 \right) & \leq 2(1) \\
+3 \left( 3x_1 - x_2 + x_3 \right) & \leq 3(3) \\
11x_1 + 5x_2 + 3x_3 & \leq 11 \\
\end{align*}
\] 

(2 - A)

Since \( x_1, x_2, x_3 \geq 0 \),

\[ \zeta = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11. \]

Hence, \( \zeta^* \leq 11 \). Now we know that \( 9 \leq \zeta^* \leq 11 \).

Instead of 2 and 3, we multiply the first constraint by \( y_1 \) and the second constraint by \( y_2 \), respectively.

\[
\begin{align*}
y_1 \left( x_1 + 4x_2 \right) & \leq y_1(1) \\
+ y_2 \left( 3x_1 - x_2 + x_3 \right) & \leq y_2(3) \\
(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + (y_2)x_3 & \leq y_1 + 3y_2. \\
\end{align*}
\] 

(2 - B)
If $y_1, y_2$ and $y_3$ satisfy the inequalities:

\[
y_1 + 3y_2 \geq 4 \\
4y_1 - y_2 \geq 1 \\
y_2 \geq 3,
\]

we have

\[
\zeta = 4x_1 + x_2 + 3x_3 \\
\leq (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + (y_2)x_3 \\
\leq y_1 + 3y_2.
\]

We now have an upper bound $y_1 + 3y_2$.

We should minimize this bound. Therefore we are led to the following optimization problem:

minimize $y_1 + 3y_2$
subject to $y_1 + 3y_2 \geq 4$

$4y_1 - y_2 \geq 1$

$y_2 \geq 3$

$y_1, y_2 \geq 0$.

The problem is called the dual linear programming problem associated with the given linear programming problem.
2. The Dual Problem (p57–)

The linear programming problem in standard form:

(5.1)

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1,2,\ldots,m \\
& \quad x_j \geq 0 \quad j = 1,2,\ldots,n,
\end{align*}
\]

and the associated dual linear program is given by

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1,2,\ldots,n \\
& \quad y_i \geq 0 \quad i = 1,2,\ldots,m.
\end{align*}
\]

(5.1) is called the primal problem.
Taking the dual of the dual returns us to the primal

The dual is rewritten as

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{m} (-b_i) y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} (-a_{ij}) y_i \leq (-c_j) \quad j = 1,2,\ldots,n \\
& \quad y_i \geq 0 \quad i = 1,2,\ldots,m.
\end{align*}
\] (5-A)

Now we can take its dual:

\[
\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{n} (-c_j) x_i \\
\text{subject to} & \quad \sum_{j=1}^{n} (-a_{ij}) x_j \geq (-b_i) \quad i = 1,2,\ldots,m \\
& \quad x_j \geq 0 \quad j = 1,2,\ldots,n,
\end{align*}
\] (5-B)

which is clearly equivalent to the primal problem (5.1).
3. The Weak Duality Theorem

Theorem 5.1. If \((x_1, x_2, ..., x_n)\) is feasible for the primal and \((y_1, y_2, ..., y_m)\) is feasible for the dual, then

\[
\sum_j c_j x_j \leq \sum_i b_i y_i
\]

Proof.

\[
\sum_j c_j x_j \leq \sum_j \left( \sum_i y_i a_{ij} \right) x_j \leq \sum_i y_i a_{ij} x_j = \sum_i \left( \sum_j a_{ij} x_j \right) y_i \leq \sum_i b_i y_i.
\]

(p59 in Textbook)

\[\text{FIGURE 5.1. The primal objective values are all less than the dual objective values. An important question is whether or not there is a gap between the largest primal value and the smallest dual value.}\]
**Optimality**

If we can exhibit a feasible primal solution \((x_1^*, x_2^*, ..., x_n^*)\) and a feasible dual solution \((y_1^*, y_2^*, ..., y_m^*)\) for which

\[
\sum_j c_j x_j^* = \sum_i b_i y_i^*, \quad (7 - A)
\]

then we may conclude that each of these solutions is optimal for its respective problem.

*Why?*

For any other feasible solution \((x_1, x_2, ..., x_n)\), we have

\[
\sum_j c_j x_j \leq \sum_i b_i y_i^* = \sum_j c_j x_j^*. \quad (7 - B)
\]

Thus, \((x_1^*, x_2^*, ..., x_n^*)\) is optimal.
4. The Strong Duality Theorem

Theorem 5.2. If the primal problem has an optimal solution,

\[ x^* = (x_1^*, x_2^*, ..., x_n^*) , \]

then the dual also has an optimal solution,

\[ y^* = (y_1^*, y_2^*, ..., y_m^*) , \]

such that

\[ \sum_j c_j x_j^* = \sum_i b_i y_i^* . \]  \hspace{1cm} (5.2)
5. Complementary Slackness

Sometimes it is necessary to recover an optimal dual solution when only an optimal primal solution is known.

Theorem 5.3. Suppose that $x = (x_1, x_2, \ldots, x_n)$ is primal feasible and that $y = (y_1, y_2, \ldots, y_m)$ is dual feasible. Let $(w_1, w_2, \ldots, w_m)$ denote the corresponding primal slack variables, and let $(z_1, z_2, \ldots, z_n)$ denote the corresponding dual slack variables. Then $x$ and $y$ are optimal for their respective problems if and only if

$$x_j z_j = 0, \text{ for } j = 1, 2, \ldots, n,$$  
$$w_i y_i = 0, \text{ for } i = 1, 2, \ldots, m.$$  

[Exercise] Prove the theorem.
8. The Dual of a Problem in General Form

The case where the linear constraints are equalities (and the variables are nonnegative):

maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, m \]
\[ \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, 2, \ldots, m \]
\[ x_j \geq 0 \quad j = 1, 2, \ldots, n. \]

We can put the problem into standard form:

maximize \[ \sum_{j=1}^{n} c_j x_j \]
subject to \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots, m \]
\[ \sum_{j=1}^{n} -a_{ij} x_j \leq -b_i \quad i = 1, 2, \ldots, m \]
\[ x_j \geq 0 \quad j = 1, 2, \ldots, n. \]
We can write down its dual.

\((y^+_i):\) the dual variables associate with the first set of \(m\) constraints

\((y^-_i):\) the dual variables associate with the second set of \(m\) constraints

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} b_i y^+_i - \sum_{i=1}^{m} b_i y^-_i \\
\text{subject to} & \quad \sum_{i=1}^{m} y^+_i a_{ij} - \sum_{i=1}^{m} y^-_i a_{ij} \geq c_j \quad j = 1, 2, \ldots, n \\
& \quad y^+_i, y^-_i \geq 0 \quad i = 1, 2, \ldots, m.
\end{align*}
\] (10 – C)

If we put \(y_i = y^+_i - y^-_i \quad (i = 1, 2, \ldots, m)\), the dual problem reduces to

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{subject to} & \quad \sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad j = 1, 2, \ldots, n.
\end{align*}
\] (10 – D)

Now the dual variables are not restricted to be nonnegative.

Equality constraints in the primal yield unconstrained variables in the dual, whereas in equality constraints in the primal yield nonnegative variables in the dual.

[Exercise] Explain why the dual of the problem (10–B) leads to the problem (10–C) and (10–D).